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KINETIC GRAIN FLOW IN A VERTICAL CHANNEL

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Abstract—A self-consistent kinetic grain flow model proposed earlier has been applied in detail to the description of rapid flow in a vertical channel. The equations of motion reduce to an ordinary differential equation for the fluctuation velocity $\bar{v}$, which is solved numerically. Boundary conditions on $\bar{v}$ are derived which incorporate the nature of grain-wall collisions. The overall flow pattern is found to depend significantly upon the grain inelasticity parameter γ ($\gamma = 0$ for elastic grains) and upon the grain diameter d. The flow velocity profile is rounded for very elastic grains and for large grains, but becomes more blunt as grain diameter decreases or γ increases. For large enough γ, a region of plug flow develops in the central region of the channel, corresponding to a vanishing grain fluctuation velocity. In this case the region of dispersed or "thermalized" grains, within which all shearing occurs, is restricted to a thin layer near each wall.

INTRODUCTION

Within the past several years there has been an increase in interest in the mechanics of systems composed of cohesionless grains (Ackermann & Shen 1982; Campbell & Brennen 1982; Cowin 1978; Haff 1983; Jenkins & Cowin 1979; Jenkins & Savage 1982; McTigue 1978; Ogawa 1978; Ogawa *et al.* 1980; Oshima 1978; Savage 1979; Savage & Jeffrey 1981; Shen & Ackermann 1982). For systems undergoing rapid shear, or where an external energy source is available, grains may be sufficiently dispersed from one another that a collisional description of grain flow constructed in analogy to the kinetic theory of gases becomes possible (Bagnold 1954; Bagnold 1956; Campbell & Brennen 1982; Haff 1983; Jenkins & Cowin 1979; McTigue 1978; Ogawa 1978; Ogawa *et al.* 1980; Savage & Jeffrey 1981).

From this point, two general approaches to the problem may be contemplated. A "realistic" model would attempt to encompass not only the rudimentary notions of momentum and energy exchange among colliding particles, but would also address the issues of grain spin, grain shape, triboelectric effects, effects of interstitial fluids, effects of velocity dependence of coefficients of restitution and friction, and so forth. From such a model, if it were soluble, one could predict the detailed behavior of actual granular systems found in nature and technology. This is an ultimate aim of the study of granular materials. At another level, however, especially in the exploratory stages of investigation which must characterize the initial development of our understanding of complicated physical systems, a simplified model which isolates and concentrates upon selected features of the system is desirable. To be useful, such models need not necessarily reproduce the detailed behavior observed in any particular experiment; rather, their utility derives from the fact that they illuminate and clarify certain features of the system which might otherwise be obscured by the complexity of a "realistic" model. It is only through the construction of such simplified models, whose inner workings are layed bare, that it is possible to decide what the next step toward "realism" should be. This is the spirit of the present work.

To implement this study, we draw upon the results of Haff (1983), who has recently given a consistent treatment of dispersed kinetic flows in terms of fluid-like equations of motion. These equations embody the requirements of mass, momentum and energy conservation in the usual way, except that the "coefficients" of viscosity and thermal diffusivity, and the collisional dissipation term in the energy equation, are not constant, but depend upon the state of the system. In principle we must also include the angular momentum equation (Haff 1983; Ohima 1978). Although the inclusion of spin would have some effect on the values of the transport functions, the overall conclusions of this paper are expected to be essentially independent of the grain–grain spin coupling. For simplicity and clarity we therefore neglect the effects of grain spin. We thus have the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \quad , \qquad [1]$$

the momentum equation in a gravitational field g,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(u_k \rho u_i) = -\frac{\partial}{\partial x_k}\left[p\delta_{ik} - \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right] + \rho g_i \quad , \qquad [2]$$

and the corresponding energy equation,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho \bar{v}^2) + \frac{\partial}{\partial x_k}[u_k \partial (\tfrac{1}{2}u^2 + \tfrac{1}{2}\bar{v}^2)] = -\frac{\partial}{\partial x_k}\left[u_k p - u_i \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \kappa \frac{\partial}{\partial x_k}(\tfrac{1}{2}\rho \bar{v}^2)\right] + \rho u_i g_i - I \quad . \qquad [3]$$

Equation [3] embodies the conduction of internal energy as "heat," and thus avoids the problems encountered by the theory of Ogawa *et al.* (1980) in describing vertical channel flow (see discussion by Jenkins & Cowin (1979)). In the above equations, ρ is the bulk density of the "granular fluid," **u** is the flow velocity, $\bar{v}$ the fluctuation velocity (henceforth called the "thermal" velocity), η the coefficient of viscosity, κ the coefficient of "thermal" diffusivity, p the pressure and I the energy dissipation term. Although by this terminology we note the analogy with thermodynamics, no purely thermodynamic results are used in the developments below.

These equations can be solved when they are supplemented by constitutive relations involving η, p, κ and I. In the kinetic model of Haff (1983) (see also McTigue 1978; Ogawa 1978; Ogawa *et al.* 1980; Savage & Jeffrey, 1981), these coefficients have the form

$$p = td\rho\frac{\bar{v}^2}{s} \quad , \qquad [4]$$

$$\eta = qd^2\rho\frac{\bar{v}}{s} \quad , \qquad [5]$$

$$\kappa = rd^2\frac{\bar{v}}{s} \quad , \qquad [6]$$

$$I = \gamma\rho\frac{\bar{v}^3}{s} \quad . \qquad [7]$$

Here s, the local mean surface-to-surface separation of grains, is related to the mass density ρ and grain diameter by

$$\rho \sim \frac{m}{(s+d)^3} \quad , \qquad [8]$$

and γ is a parameter measuring the inelasticity of grain collisions. These relations were derived under the assumption that the constituent grains always remain closer together than a grain diameter ($s << d$), and only flows which satisfy this criterion are considered here.

γ is equal to $b(1 - \epsilon^2)$, where ϵ, the coefficient of restitution, ranges between 1 (completely elastic) and 0 (completely inelastic) and b is a constant. If the particle distribution functions were known, the quantities t, q, r and b could be evaluated exactly. Here, however, our aim is to elucidate the flow characteristics of the kinetic grain model in a vertical channel in a semiquantitative manner. In real flows unavoidable, if small, effects of Coulomb friction, grain spin and grain irregularities, together with uncertainties in the details of grain–wall collisions, seem to make excessive precision in specification of the collisional model self-deceiving. Therefore we simply take t, q, r and b as dimensionless constants. They are of order unity. This procedure has the added advantage that the contributions of pressure and of viscous forces (t and q), of conduction (r) and the effects of grain inelasticity (b) can often be readily identified at a glance in analytic expressions.

Although [1]–[3] superficially resemble the equations of motion of a classical fluid, the auxiliary relations, [4]–[7] show that within the model the pressure, viscosity and other quantities characterizing the state of the granular fluid are determined entirely by the thermal velocity $\bar{v}$ and the density parameter s. These in turn are strongly coupled to the flow velocity through the equations of motion. Perceived in this way, it is clear that a self-consistent treatment of dispersive ($s > 0$) grain flow is a necessity; quantities such as thermal diffusivity and viscosity are not provided *a priori*, but must be determined as part of the solution.

SOLVING THE EQUATIONS OF MOTION

As shown by Haff (1983) for steady two-dimensional flows, the equations of motion, [1]–[3], become linear in $\mathbf{u}$ and $\mathbf{v}$. Furthermore, with suitable substitution one can generally reduce the problem to the solution of a single ordinary differential equation. Earlier analytical work (Haff 1983) gave explicit solutions for the flow velocity and thermal velocity in horizontal Couette flow. In order to obtain simple analytic solutions, either the gravitational field $\mathbf{g}$ had to be set equal to zero, or the grain inelasticity γ had to be set equal to zero. In the former case, the velocity fields are expressible in terms of trigonometric functions of position within the Couette channel, and in the latter case as trigonometric functions of the logarithm of position.

It is clear that gravitational effects are always present to a greater or lesser degree in any terrestrial experimental configuration, but it is less clear how important the role of grain inelasticity might be. In fact, the presence of even a tiny amount of inelasticity in grain–grain collisions will have a large effect on the overall flow and thermal patterns if a large enough number of grains are involved. This is because energy loss through collisions is an exponentiating process, so that energy supplied at one point, either internally through viscous losses, or externally through a boundary, is rapidly dissipated within a characteristic e-folding length λ proportional to the grain diameter (see [18] below). If an experimental flow apparatus under consideration has a linear dimension which is large compared to λ, as is often the case even for very elastic grains, then it is important to include explicity the effects of inelasticity.

In order to study dispersive grain flow in an approximation to a physically realizable configuration, we have looked at solutions to [1]–[7] for the case of flow in a long, two-dimensional channel, figure 1. The gravitational force is parallel to the channel walls and grain–grain collisions are characterized by the inelasticity parameter γ (see [7]).

The steady-state vertical channel problem, including the effects of both gravity and inelasticity, can be solved in the following way. Because of the condition $s << d$, the density ρ can be taken essentially constant. The x and y-components of the momentum equation then give the dimensionless pressure

$$P_0 = \frac{p_0}{\rho g h} \quad , \qquad [9]$$

where p_0 is a constant, i.e. the pressure is the same everywhere in the channel, and

$$\Sigma(y) = -Y + \Sigma_0 \quad , \qquad [10]$$

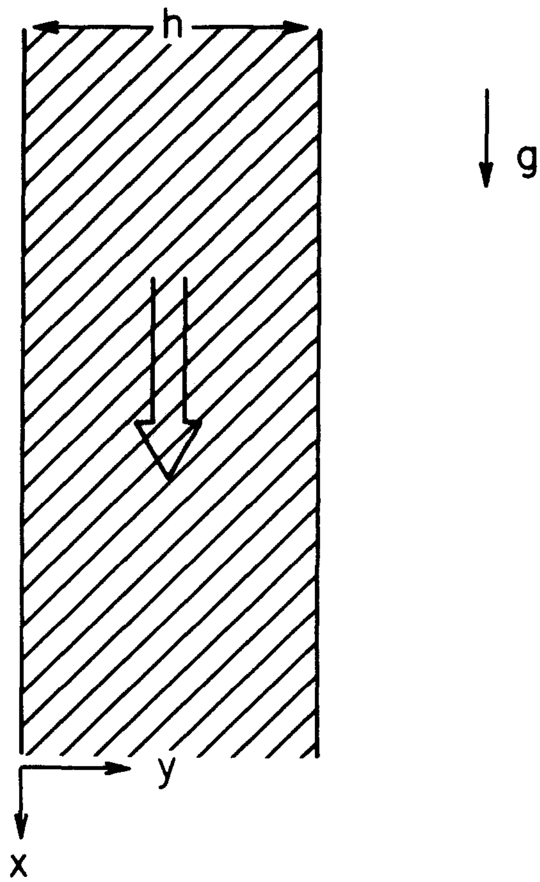

Figure 1. Schematic of vertical channel.

where $Y = y/h$, $\Sigma_0 = \sigma_0/\rho hg$, and σ_0 is a constant shear stress. After some manipulation, using [4] and the relation $\sigma = \eta\, du/dy$, [3] can be reduced to a linear (but not linearized) equation for V,

$$\frac{d^2V}{dY^2} + \frac{t}{r}\frac{1}{\delta^2}\left[\frac{t}{q}\left(\frac{\Sigma_0 - y}{P_0}\right)^2 - \frac{\gamma}{t}\right]V = 0 \quad , \qquad [11]$$

where $\delta = d/h$ and $V = \bar{v}/(gh)^{1/2}$.

The solutions to this equation are the parabolic cylinder functions (for the special case where $\gamma = 0$ these reduce to Bessel functions of order ¼). Jenkins & Savage (1982) have independently derived an equation for vertical flow of the form given in [11], but did not discuss the solutions.

Explicit solutions may be determined once suitable boundary conditions have been specified. A general treatment of boundary conditions for the kinetic model of granular matter has recently been given by Hui *et al.* (1984). In the present case we can argue as follows. If e_w is the coefficient of restitution for a grain–wall collision, then, from arguments similar to those made in (Haff 1983), the rate of energy loss to the wall is

$$Q_w = a\tfrac{1}{2}m\bar{v}^2(1 - e_w^2)\frac{1}{d^2}\frac{\bar{v}}{s} \quad , \qquad [12]$$

where a is a dimensionless constant of order unity. If this is equated to the energy delivered to the wall via conduction,

$$Q = \kappa\frac{d}{dy}(\tfrac{1}{2}\rho\bar{v}^2) \quad , \qquad [13]$$

then we obtain the desired boundary condition on $\bar{v}$

$$\bar{v}_0 = \frac{2rd}{a(1 - e_w^2)} \frac{d\bar{v}_0}{dy} \quad , \qquad [14]$$

where the subscripts indicate that the thermal velocity and its derivative are to be evaluated at the wall. A similar equation can be derived for the flow velocity u by a consideration of the stresses applied to the wall via grain–wall collisions. For simplicity, and without loss of generality, we considered only the case of a channel with rough walls, for which the no-slip condition $u(0) = u(h) = 0$ is appropriate.

To solve [11] we set $t = q = r = a = 1$. For problems in which the two walls of the channel are identical (the case considered here), the appropriate solution to [11] is symmetrical about the midline ($Y = ½$) of the channel. In this case, $\Sigma_0 = ½$. Equation [11] is integrated numerically for an arbitrary choice of the pressure P_0 and an arbitrary value of $V_0 = \bar{v}_0/(gh)^{1/2}$, subject to [14] and the condition $dV/dY = 0$ at $Y = ½$. If the latter condition cannot be satisfied, a new value of P_0 is chosen, keeping V_0 fixed, and [11] reintegrated. This procedure continues until a suitable value of P_0 is determined.

Because [11] and [14] are linear and homogeneous, they do not determine the normalization of V. To do so requires an additional condition. This condition can be expressed (Haff 1983) in terms of the "free space" parameter, Δh, which is the thickness of the empty space which would be produced if all the grains in the channel were uniformly compressed to maximum density ($s = 0$) against one wall. It is easy to show that

$$\Delta H = \frac{3}{\delta} \int_0^1 S(Y) dY \quad , \qquad [15]$$

where $\Delta H = \Delta h / h$ and $S = s / h$. From this equation, and [4] for the pressure, it follows that P_0 and V must be related by

$$P_0 = \frac{3t}{\Delta H} \int_0^1 V^2 \, dY \quad , \qquad [16]$$

which determines the magnitude of V.

Finally, from [10] and the fact that the stress σ is related to the flow velocity by $\sigma = \eta \, du/dy$, the flow velocity $U = u/(gh)^{1/2}$ can be calculated from

$$U(Y) = \frac{t}{q\delta} \int_0^Y \left(\frac{\Sigma_0 - Y}{P_0} \right) V(Y) dY \quad , \qquad [17]$$

which reflects the no-slip boundary condition.

SOLUTIONS TO THE EQUATIONS OF MOTION

Figure 2 shows thermal and flow velocities V and U computed for a grain diameter to channel width ratio $\delta = 0.01$ and free space parameter $\Delta H = 0.1$. In this example the grain–grain inelasticity coefficient is $\gamma = 0.01$, and the grain–wall coefficient of restitution e_w is 0.66. From the shape of the curve for U it is clear that most of the shearing, and hence viscous heating, occurs away from the center of the channel. The damping length

$$\Lambda \equiv \lambda / h \sim \delta / \gamma^{1/2} \qquad [18]$$

which is a measure of the distance over which a pulse of granular heat can be conducted before dissipation is on the order of 0.1, rather less than the channel width, and consequently the curve of thermal velocity exhibits a central dimple. For grains with nearly perfect elasticity ($\gamma \approx 0$), the central dimple disappears, while for increasingly inelastic grains the

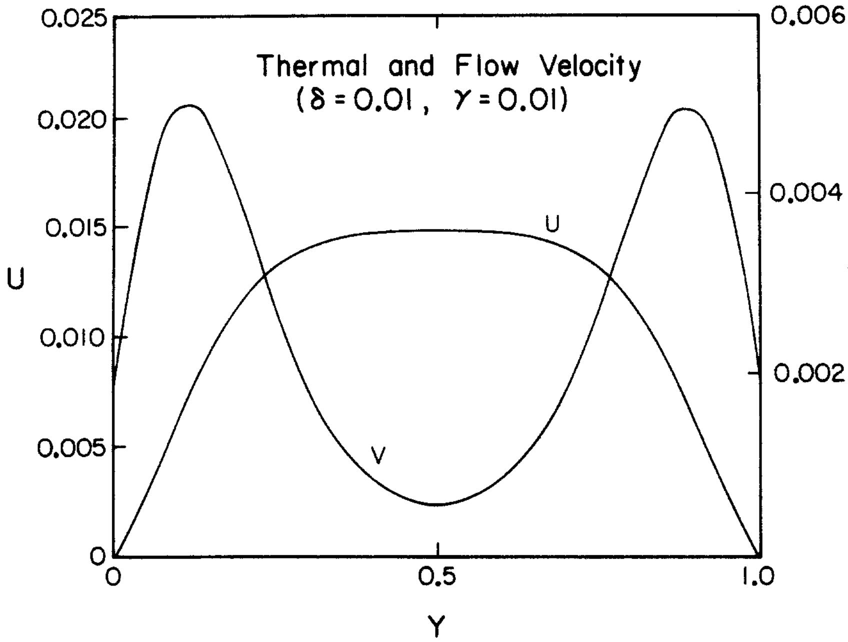

Figure 2. Thermal and flow velocity profiles for $\gamma = 0.01$, $\delta = 0.01$, $\Delta H = 0.1$ and $e_w = 0.66$.

dimple rapidly deepens until the thermal velocity vanishes in the central region of the channel. In view of the fact that the damping length is scaled by the grain diameter, inelastic effects will be more important in systems containing small grains than in a comparably sized system filled with larger grains. This implies that kinetic systems of very fine grains (true powders) would in the absence of pneumatic and cohesive effects prefer to deform along localized shear zones (although in most real powders such effects are important).

Figure 3 shows the behavior of the thermal velocity for $\gamma = 0.1$. At this value of the inelasticity parameter energy cannot be conducted efficiently into the center of the channel and the granular temperature falls nearly to zero here. The various curves in figure 3 correspond to different choices of the wall coefficient e_w. The more elastic the grain–wall

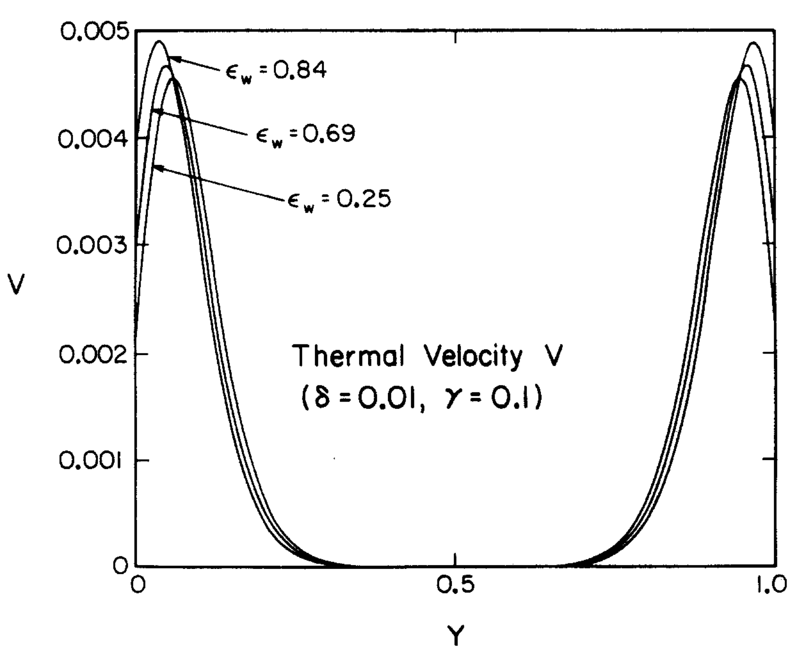

Figure 3. Thermal velocity profiles for $e_w = 0.25$, 0.69 and 0.84 with $\gamma = 0.1$, $\delta = 0.01$ and $\Delta H = 0.1$

collisions are, the greater the value of V_0, as expected, but the general shape of the thermal velocity profile is not sensitive to the boundary condition on V.

Corresponding to the change in the thermalization pattern with increasing γ is a shift in the flow velocity profile, figures 2 and 4. At small values of γ, the flow profile is rounded, rising gently to a maximum in the center of the channel, figure 2. As γ increases, the flow profiles become increasingly blunt, until for values of γ larger than about 0.1, in the case considered here, the flow pattern becomes so flat that material in the inner part of the channel travels essentially as an undeforming plug (figure 4).

The present model is predicated upon the hypothesis of binary collisions, and this hypothesis fails in the central channel region when $V \to 0$. Nonetheless, the solutions obtained in the peripheral shear bands remain semiquantitatively correct. Toward the channel center bulk grain density increases as the grains crowd together (figure 5) in an attempt to sustain pressure in response to falling thermal velocity. Increasing bulk densities correspond to increasing shear resistance; at sufficiently high density no shearing at all is possible. Moreover, for any flow law in a symmetrical channel the center of the channel is a zone of low applied shear. Plug formation, therefore, is practically inescapable.

The velocity and density solutions obtained here are not strongly dependent on where the "inner" boundary condition is applied. We have used $\mathrm{d}V/\mathrm{d}Y = 0$ at $Y = \frac{1}{2}$ but since the gradient vanishes to a good approximation over most of the width of the plug (figure 3) we can apply a similar boundary condition near the plug boundary to obtain similar solutions in the shear bands. Therefore, our results are not invalidated by the failure of kinetic model within the plug zone, where the thermal velocity vanishes. A more complete treatment of grain–grain frictional effects will eventually allow a more accurate description of the transition region between the zone of dispersed grains and the zone of consolidated grains.

Observations of plug flow in the central region of a vertical channel have been reported by several authors (Nedderman & Laohakul 1980; Savage 1979; Shigeki 1970; Takahashi & Yanai 1973). Not all of the experimental configurations conformed exactly to the free flowing two-dimensional regime treated in this paper, nor is it clear that shear rates were always sufficiently high to ensure the validity of the kinetic picture even in the shear zones. Nonetheless, the qualitative agreement between theory and experiment is interesting. In particular, Savage's experiments (Savage 1979) showed that while for narrowly spaced walls (approximately 30 grain diameters apart) the shear region filled the whole channel, for a wider channel central plug flow regions could exist. In the latter case the shear-flow regions

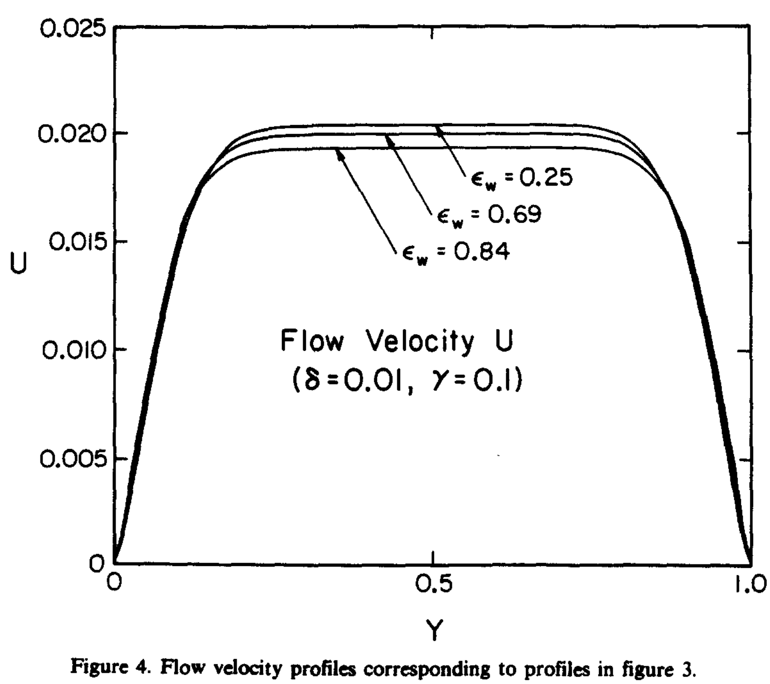

Figure 4. Flow velocity profiles corresponding to profiles in figure 3.

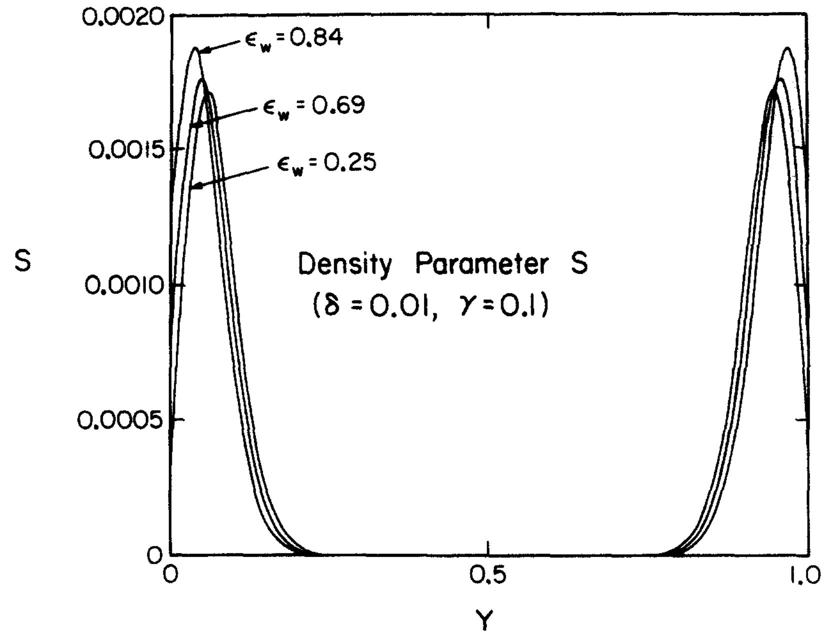

Figure 5. Density parameter profiles corresponding to profiles in figure 3.

were restricted to boundary layers near the walls. Moreover, the width of the shear zones was found to be proportional to the particle diameter, as expected on the basis of the damping length. [18]. Experimental flow-visualization photographs showing velocity profiles and plug flow zones similar to those discussed here can be found in Savage (1984, p. 307).

Figure 6 illustrates the thermal velocity $V(e_w = 0.62)$ for a dimensionless grain diameter $\delta = 0.05$ and $\gamma = 0.1$. Because these grains are so large and because the thickness of the shear layer scales with grain diameter, energy can be conducted to the central part of the channel, unlike in the case of the smaller grains (figure 2). Figure 6 also shows the flow velocity as a function of position. The profile is generally rounded to somewhat blunt. Of course if the channel were made considerably wider, keeping the grain size the same, then again a region of plug-flow would occur in the central part of the channel.

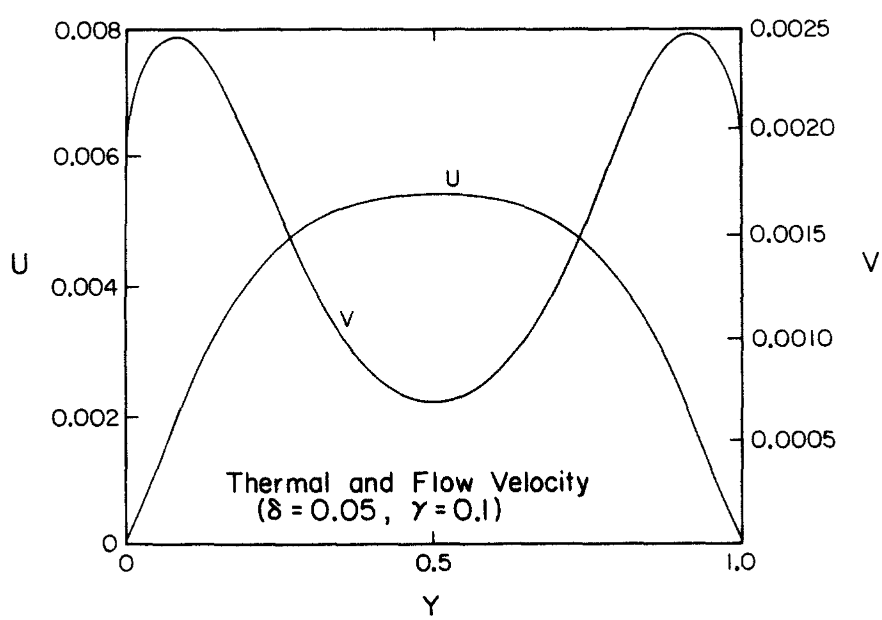

Figure 6. Thermal and flow velocity profiles for grains of dimension's diameter $\delta = 0.05$, $\Delta H = 0.1$, $e_w = 0.62$ and $\gamma = 0.1$.

The narrowness of the flow channel in some of these experiments (Savage 1979) and the narrowness of the shear zones derived in some cases from the theoretical model raise concern about the validity of the continuum hypothesis, on which the kinetic picture is based. A continuum description of a granular system can certainly never be as good as a continuum description of a molecular system. The differential equations used to describe grain flow here are dimensionally correct, however, and therefore in essence describe granular systems at the dimensional analysis level even for systems containing only several grains. The equations of course are not especially accurate at this level. A continuum picture will usually be useful if the dependent variables of the theory do not change radically over distances less than the diameter of a single particle.

SUMMARY

Within the kinetic model a simple and definite picture of vertical channel flow emerges. The shape of the thermal and flow velocity profiles is influenced profoundly by the degree of grain–grain inelasticity, and no treatment which neglects the effects of this energy loss process can hope to give an adequate description of the flow. The appropriate boundary values on both the thermal and flow velocities can be formulated succinctly in terms of collision rates and the nature of the grain–wall interaction. The overall character of the flow pattern, however, is not strongly dependent upon the inelasticity of the grain–wall collision.

Perhaps the most characteristic feature of the flows studied here is the restriction of the principal region of shearing to two relatively narrow bands near the sides of the channel. For all except the most elastic grains (or narrow channels), the central region of flow travels as an essentially undeforming plug. The fact that the kinetic model is invalid within the plug region does not alter the validity of the solutions within the shear zones.

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